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Turbulent natural convection in a horizontal fluid layer with volumetric energy sources: an intermediate layer

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Abstract—The turbulent natural convection driven by volumetric energy sources in a horizontal fluid layer bounded by a lower adiabatic plate and an upper isothermal plate, is studied at large values of the internal Rayleigh number. The existence of an intermediate layer is identified, the length scale of which is $Re^{-3/7}$ (Re is a Reynolds number based on the outer characteristic velocity and length scales) rather than $Re^{-1/2}$ proposed by Long and Chen. The maxima in Reynolds heat flux and in turbulent energy dissipation are associated with the intermediate layer. The intermediate layer is matched with the inner and outer layers. It is shown that to the lowest order, the two-layer classical theory should suffice and to the next order an intermediate layer is needed.

1. INTRODUCTION

THE TURBULENT natural convection driven by volumetric heat sources is of interest in the convection processes in the earth's mantle [1–5] and nuclear power reactor safety; specifically, heat removal from a molten pool of fuel and reactor material in the event of a core melt-down accident [6].

The problem of natural convection in a volumetrically heated fluid layer confined between two horizontal plates has attracted the attention of several workers. The flow visualization in the post-stability regime, for Rayleigh numbers from transition to turbulence, has been conducted in refs. [7–8]. In the turbulent regime, the surface heat transfer measurements have been reported in refs. [9–11].

The theoretical studies for turbulent natural convection are based on a two-layer (inner and outer) classical theory. In the inner layer, near the surface, molecular conductive-viscous transport is important and in the outer layer, away from the surface, eddy transport dominates over molecular transport. The laboratory measurements, on the whole, are not favourable with similarity theory and it is generally regarded that Rayleigh numbers in these experiments are not as high as theory may require [12–14].

The convection due to volumetric heat sources has been theoretically studied by Fiedler and Wille [15] and Cheung [16, 17]. In a recent attempt Cheung [17] analysed the classical two layers and matched them to get the heat transfer law. In ref. [17] the inner layer arguments are based on the equations of individual fluctuating components. The arguments of mean motion, however, should not be based on the dynamics of individual fluctuations. It is well known that in wall bounded turbulent (forced convection) shear flows, the equations of individual fluctuations are not invoked in obtaining the logarithmic law [13, 14]. Further, in the outer layer [17] a mixing length closure is adopted, that directly gives the stipulated power law, in effect, assuming what should be a major deduction. The work of Cheung [17] also disregards Millikan's argument, which is perfectly valid from the singular perturbation viewpoint [18].

The earlier analysis of Cheung [16] based on the mixing length closure hypothesis shows that the maximum Reynolds heat flux lies in the inner layer, the length scale of which is $LRa_I^{-1/4}$. By extending the arguments of ref. [17] or otherwise it can be shown that the maximum Reynolds heat flux lies in a domain, the length scale of which is $LRa_I^{-1/7}$, different from classical inner and outer length scales. In terms of the order of magnitude, the maximum is associated with a layer that is intermediate between the classical inner and outer layers.

The main aim of this work is to analyse the intermediate layer for heat source driven thermal convection in a horizontal fluid layer bounded by a lower adiabatic plate and an upper isothermal plate. The method of matched asymptotic expansions is used and the internal Rayleigh number, Ra_I, is regarded as large. Based on the under-determined equations of the mean temperature profile and the mean turbulent energy dissipation, the existence of an intermediate layer is demonstrated, that is associated with maxima in Reynolds heat flux and in turbulent energy dissipation. The asymptotic expansions in the three (inner, intermediate and outer) layers are matched in the two overlap domains. It is found that under a certain condition, the lowest order results are the same as given by the two-layer classical theory; the intermediate layer forms the matching domain between the inner and outer layers and there is no need to treat it separately. Thus to the lowest order, the two-layer classical theory should suffice. To the next order the intermediate layer is needed and the flow field should comprise of three distinct layers.

In a related problem of Rayleigh–Bénard convection

NOMENCLATURE

A	slope of $-1/3$ temperature profile in the classical theory
A_i	slope of $-1/3$ temperature profile in O_i
A_o	slope of $-1/3$ temperature profile in O_o
$C, \tilde{C}$	constants
C_p	specific heat
e_1, e_2, e_3	gauge functions in the expansions for inner, outer and intermediate layers representing the higher order effects defined by equation (84)
$f, \mathcal{F}, F$	inner, intermediate and outer variables for mean temperature
g	gravity acceleration
$h, \mathcal{H}, H$	inner, intermediate and outer variables for Reynolds heat flux
k	unit vector in vertical direction
L	normal distance between the two horizontal plates
M	mean kinetic energy of velocity fluctuations
N	mean transport of kinetic and potential energies of fluctuations in the normal direction
Nu	Nusselt number
O_i	overlap region between inner and intermediate layers
O_o	overlap region between outer and intermediate layers
P	instantaneous pressure
p	fluctuations in pressure
q	appropriate Reynolds heat flux, $\overline{w\phi}$
q_m	maximum value of Reynolds heat flux
q_w	heat transfer at upper hot plate
Ra	Rayleigh number
Ra_I	internal Rayleigh number, $g\beta\phi L^5\sigma^2/\nu^3$
Re	characteristic Reynolds number, u_oL/ν
S	strength of uniform heat source distribution
t_i, t_δ, t_o	scales of Reynolds heat flux in inner, intermediate and outer layers
T	instantaneous temperature
$\bar{T}$	mean temperature
T_w, T_r, T_o	reference temperatures in inner, intermediate and outer layers
$\mathbf{u}$	instantaneous velocity vector
u_i, u_δ, u_o	scales for velocity fluctuations in inner, intermediate and outer layers
u_τ	friction velocity
w	instantaneous normal velocity component
y	normal distance from the upper hot plate measured downwards
Y	outer variable
Y_l	levelling off point of mean temperature profile
Y_m	location of maximum in tangential R.M.S. velocity
Z	normal coordinate measured upwards from the lower insulated plate.

Greek symbols

α	constant
β	coefficient of thermal expansion
β_1, β_2	constants
δ	thickness of the intermediate layer, $LRe^{-3/7}$
ε	mean energy dissipation
ε_m	maximum value of ε
$\varepsilon_i, \varepsilon_\delta, \varepsilon_o$	mean dissipation in the inner, intermediate and outer layers
ζ	intermediate variable
ζ_m	location of maxima in Reynolds heat flux
$\zeta_{m\varepsilon}$	location of maxima in turbulent energy dissipation
η	inner variable
ν	molecular kinematic viscosity
ρ	mean density of the fluid
σ	molecular Prandtl number
ϕ	non-dimensional heat source strength, $\bar{S}/\rho_o C_p$.

between two plates (where the lower plate is hot and the upper plate is cold) a mesolayer (or intermediate layer) has been proposed by Long and Chen [19]. Based on an assumed analogy with forced convection, the length scale of the intermediate layer, for natural convection flows is proposed [19] as $Re^{-1/2}$ (where Re is a Reynolds number based on the outer characteristic velocity and length scales). As shown in Section 5, the analogy between the forced and natural convection flow is not as simple as postulated in ref. [19]. The present work shows that the length scale of the intermediate layer is $Re^{-3/7}$. Although the scatter in the data, presented in Section 5, is too large to distinguish $-3/7$ from $-1/2$, the fact remains that the present work is more rational and removes some of the inconsistancies of the classical theory.

2. GOVERNING EQUATIONS

Let a volumetrically heated fluid be bounded between two horizontal rigid plates, where the lower plate is adiabatic and the upper plate is isothermal. The volumetric heat generation is assumed to be spatially uniform and statistically steady one-dimensional (1-D) transport is maintained. The governing equations of instantaneous motion, under the Boussinesq approximation are

$$\nabla \cdot \mathbf{u} = 0, \qquad (1)$$

$$\partial \mathbf{u}/\partial t+\mathbf{u}\cdot\nabla\mathbf{u} = -\rho_o^{-1}\nabla P+\nu\nabla^2\mathbf{u}+\hat{k}g\beta T, \quad (2)$$

$$\partial T/\partial t+\mathbf{u}\cdot\nabla T = \nu\sigma^{-1}\nabla^2 T+S/\rho_o C_p. \quad (3)$$

For statistically stationary and 1-D flow the dependent variables may be decomposed into mean and fluctuating parts

$$T = \bar{T}+\theta, \quad S = \bar{S}+S', \quad \bar{T} = \bar{T}(Z),$$
$$\bar{\theta} = \bar{u} = \bar{S'} = 0, \quad (4)$$

where Z is the vertical coordinate measured upwards from the lower surface. The equation for the mean temperature distribution obtained from equations (3) and (4) is

$$\frac{\nu}{\sigma}\frac{\mathrm{d}^2\bar{T}}{\mathrm{d}Z^2}-\frac{\mathrm{d}q}{\mathrm{d}Z}+\phi = 0, \quad (5)$$

where

$$q = \overline{w\theta}, \quad \phi = \bar{S}/\rho_o C_p. \quad (6)$$

The equations for the fluctuating component are

$$\frac{\partial}{\partial t}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u} = \nu\nabla^2\mathbf{u}-\rho_o^{-1}\nabla p+\hat{k}g\beta\theta, \quad (7)$$

$$\frac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta = \frac{\nu}{\sigma}\nabla^2\theta+\nabla\cdot(\overline{\mathbf{u}\theta})-w\frac{\partial\bar{T}}{\partial Z}+\frac{S'}{\rho_o C_p}, \quad (8)$$

where $p = P-\bar{P}-\rho_o\bar{w}^2$ is the fluctuating pressure and w is the fluctuating velocity component in the vertical direction. Equation (5) for the mean temperature profile is integrated once to give

$$\frac{\nu}{\sigma}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}+q = \phi L(1-y/L), \quad (9)$$

$$y = L-Z, \quad (10)$$

where y is the normal coordinate measured downwards from the upper surface and L is the distance between the plates. In writing down equation (9) the boundary condition of the lower adiabatic plate has been used to eliminate the constant of integration. The heat flux, Ω, at the upper surface

$$\Omega = \phi L, \quad (11)$$

is assumed known. The temperature difference, ΔT, between the top and bottom plates is merely a consequence of the volumetric heating of the layer.

In the outer layer the molecular transport is insignificant and the appropriate variables are L, Ω and $g\beta$. The outer length scale is L. The outer scales for velocity, u_o, and temperature, t_o, based on these parameters may be written as

$$u_o = (g\beta\Omega L)^{1/3}, \quad t_o = \Omega/u_o. \quad (12)$$

For the inner layer the molecular transport is significant and the flow is expected to depend on ν, Ω, $g\beta$ and σ. For σ of order unity, the inner scales of velocity, u_i, and temperature, t_i, from dimensional arguments may be written as

$$u_i = (\nu g\beta\Omega)^{1/4}, \quad t_i = \Omega/u_i. \quad (13)$$

The inner length scale is ν/u_i. These scales are consistent with ref. [17] when the molecular Prandtl number, σ, is of order unity.

3. THREE LAYER ANALYSIS

The under-determined equation (9) for the mean temperature distribution is analysed by the method of matched asymptotic expansions. The flow field is divided in three (inner, intermediate and outer) layers that are matched in the two overlap domains.

3.1. *Inner region*

Based on the scales defined in Section 2, the inner variables can be taken as

$$\eta = yu_i/\nu,$$
$$\bar{T} = T_w+t_i f(\eta), \quad q = u_i t_i h(\eta), \quad (14)$$
$$t_i = \phi L/u_i, \quad u_i = (\nu g\beta\phi L)^{1/3}. \quad (15a,b)$$

In terms of inner variables, equation (9) produces

$$\sigma^{-1}\frac{\mathrm{d}f}{\mathrm{d}\eta}+h = 1-Re^{-3/4}\eta, \quad (16)$$

where Re is a Reynolds number based on the outer characteristic velocity and the length scale, defined as

$$Re = u_o L/\nu. \quad (17)$$

The inner expansions can be written as

$$f = f_1+e_1 f_2+,\ldots, \quad (18a)$$
$$h = h_1+e_1 h_2+,\ldots, \quad (18b)$$

where e_1 is the gauge function. The lowest order equation

$$\sigma^{-1}\frac{\mathrm{d}f_1}{\mathrm{d}\eta}+h_1 = 1, \quad (19a)$$

shows that the total heat flux remains constant. The next order equation is

$$\sigma^{-1}\frac{\mathrm{d}f_2}{\mathrm{d}\eta}+h_2 = 0. \quad (19b)$$

3.2. *Outer layer*

The outer variables are

$$Y = y/L,$$
$$\bar{T} = T_o+t_o F(Y), \quad q = u_o t_o H(Y), \quad (20)$$
$$t_o = \phi L/u_o, \quad u_o = (g\beta\phi L^2)^{1/3}. \quad (21a,b)$$

The equation for the mean temperature distribution, equation (9), in terms of outer variables, may be written as

$$H+(\sigma Re)^{-1}\frac{\mathrm{d}F}{\mathrm{d}Y} = 1-Y. \quad (22)$$

The outer expansions are

$$F = F_1+e_2 F_2+,\ldots, \quad (23a)$$
$$H = H_1+e_2 H_2+,\ldots, \quad (23b)$$

where e_2 is the gauge function. The equations connecting the various order terms are

$$H_1 = 1 - Y, \tag{24a}$$

$$H_2 + (e_2\, Re\, \sigma)^{-1}\frac{\mathrm{d}F_1}{\mathrm{d}Y} = 0. \tag{24b}$$

3.3. *Intermediate layer*

Let the length scale of the intermediate layer be δ, the velocity scale be u_δ and temperature scale be t_δ. Without loss of generality, one can assume that the product of the scales for velocity and temperature is of the order of the heat flux at the wall, i.e.

$$u_\delta t_\delta = \phi L. \tag{25}$$

Let the temperature distribution in the intermediate layer be

$$\bar{T} = T_r + t_\delta \mathscr{F}(\zeta), \quad \zeta = y/\delta, \tag{26}$$

where T_r is some reference temperature in the intermediate layer. Substituting equation (26) in equation (9) for the mean temperature profile one obtains

$$\frac{q}{\phi L} - 1 + \frac{\delta}{L}\left[\zeta + \frac{\nu}{u_\delta L}\left(\frac{L}{\delta}\right)^2 \sigma^{-1}\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\zeta}\right] = 0. \tag{27}$$

The terms in square brackets representing molecular conduction and heat generation are of equal order for

$$\delta/L = (\nu/u_\delta L)^{1/2}. \tag{28}$$

Introducing the outer variable for Reynolds heat flux

$$q/\phi L = 1 + \left(\frac{\nu}{u_\delta L}\right)^{1/2}\mathscr{H}(\zeta), \tag{29}$$

in equation (27) one obtains

$$\sigma^{-1}\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\zeta} + \mathscr{H} = -\zeta. \tag{30}$$

The intermediate expansions may now be written as

$$\mathscr{F} = \mathscr{F}_1 + e_3\mathscr{F}_2 + ,\ldots, \tag{31a}$$

$$\mathscr{H} = \mathscr{H}_1 + e_3\mathscr{H}_2 + ,\ldots, \tag{31b}$$

where e_3 is the gauge function. The equations governing the various order terms are

$$\sigma^{-1}\frac{\mathrm{d}\mathscr{F}_1}{\mathrm{d}\zeta} + \mathscr{H}_1 = -\zeta, \tag{32a}$$

$$\sigma^{-1}\frac{\mathrm{d}\mathscr{F}_2}{\mathrm{d}\zeta} + \mathscr{H}_2 = 0. \tag{32b}$$

The intermediate expansions are singular at the wall as the condition of zero Reynolds heat flux can not be satisfied. So far u_δ is unspecified and will be determined from the matching of the intermediate layer with the inner and outer layers described in the next section.

3.4. *Matching*

In the three layers the scales of velocity and temperature satisfy the relation

$$u_i t_i = u_\delta t_\delta = u_o t_o = \Omega. \tag{33}$$

In the presence of the intermediate layer, the matchability postulate of the classical inner and outer layers should be relaxed. Therefore, the matchability requirements of the intermediate layer can be considered with the inner and outer layers. The matching of the temperature profile in the inner and intermediate layers (say, overlap domain O_i) requires

$$T_w + t_i f_1(\eta \to \infty) \sim T_r + t_\delta \mathscr{F}_1(\zeta \to 0). \tag{34}$$

This is a functional equation [18] whose solution can be obtained by following Millikan's argument. Differentiating relation (34) with respect to Y, and using equation (33), one obtains

$$\frac{\partial f_1}{\partial \eta}(\eta \to \infty) \sim \left(\frac{\nu}{Lu_\delta}\right)^{1/2}\frac{\partial \mathscr{F}_1}{\partial \zeta}(\zeta \to 0). \tag{35}$$

Based on equations (11) and (33) the square root in equation (35) is estimated as

$$\left(\frac{\nu}{Lu_\delta}\right)^{1/2} = (\zeta/\eta)^{4/3}\frac{(g\beta\phi L\delta)^{1/3}}{u_\delta}, \tag{36}$$

and relation (35) becomes

$$\lim_{\eta\to\infty}\eta^{4/3}\frac{\partial f_1}{\partial \eta} \sim \frac{(g\beta\phi L\delta)^{1/3}}{u_\delta}\lim_{\zeta\to 0}\zeta^{4/3}\frac{\partial \mathscr{F}_1}{\partial \zeta}. \tag{37}$$

The matching of relation (37) requires

$$u_\delta = (g\beta\phi L\delta)^{1/3}, \tag{38}$$

which provides an additional relation needed for determination of δ and u_δ. The matching relation (37) now becomes

$$\lim_{\eta\to\infty}\eta^{4/3}\frac{\partial f_1}{\partial \eta} \sim \lim_{\zeta\to 0}\zeta^{4/3}\frac{\partial \mathscr{F}_1}{\partial \zeta}. \tag{39}$$

This relationship is of the type $N(\eta) \sim M(\zeta)$ each side of which must approach a constant (say, $3A_i$) independent of η and ζ. Integration of each side leads to

$$f_1 \sim -A_i\eta^{-1/3} + B, \quad \eta \to \infty, \tag{40a}$$

$$\mathscr{F}_1 \sim -A_i\zeta^{-1/3} + C, \quad \zeta \to 0, \tag{40b}$$

where B and C are constants of integration. Using relation (40) the matching of the temperature profile from relation (34) yields

$$T_w + t_i B = T_r + t_\delta C. \tag{41}$$

Similarly, the matching of the Reynolds heat flux in the inner and intermediate layers leads to

$$h_1 \sim 1 - \tfrac{1}{3}A_i\eta^{-4/3}, \quad \eta \to \infty, \tag{42a}$$

$$\mathscr{H}_1 \sim -\zeta - \tfrac{1}{3}A_i\zeta^{-4/3}, \quad \zeta \to 0. \tag{42b}$$

Likewise the matching of the temperature profile and the Reynolds heat flux in the overlap domain O_o of the intermediate and outer layers gives

$$\mathscr{F}_1 = -A_o\zeta^{-1/3} + \tilde{C}, \quad \zeta \to \infty, \tag{43a}$$

$$F_1 = -A_o Y^{-1/3} + D, \quad Y \to 0, \tag{43b}$$

$$T_r + t_\delta \tilde{C} = T_o + t_o D, \tag{44}$$

$$\mathscr{H}_1 = -\zeta - \tfrac{1}{3}A_o\zeta^{-4/3}, \quad \zeta \to \infty, \tag{45a}$$

$$H_1 = 1 - Y, \quad Y \to 0, \tag{45b}$$

3.6. *Heat transfer law*

The length and velocity scales in the intermediate layer can be obtained from equations (28) and (38) as

$$\delta/L = Re^{-3/7}, \quad u_\delta/u_o = Re^{-1/7}. \tag{46}$$

Eliminating T_r between equations (41) and (44) one obtains

$$(T_o - T_w)/t_i = B-(C-\tilde{C})t_\delta/t_i - Dt_o/t_i. \tag{47}$$

Based on equation (33), relation (47) can be written as the heat transfer law

$$Nu = (\sigma^2 Ra_I)^{1/4}/[B-(C-\tilde{C})(Ra_I/\sigma^2)^{-1/28} - D(Ra_I/\sigma^2)^{-1/12}], \tag{48}$$

where B, C and $\tilde{C}$ are functions of σ. Here Ra_I is the internal Rayleigh number, defined by

$$Ra_I = \sigma^2 g\beta\phi L^5/\nu^3. \tag{49}$$

4. TURBULENT MEAN KINETIC ENERGY EQUATION

The equation involving the mean turbulent kinetic energy can be obtained by multiplying equation (7) with $\mathbf{u}$ and taking averages of the resultant equation as

$$\varepsilon = g\beta q + \frac{dN}{dy} + \nu\frac{d^2M}{dy^2}, \tag{50}$$

$$\varepsilon = \langle \nu(\partial u_i/\partial x_j)^2\rangle, \quad q = \langle w\theta\rangle, \tag{51a,b}$$

$$N = \langle w(p/\rho_o + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u})\rangle, \quad M = \langle \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\rangle. \tag{51c,d}$$

Here ε is the mean energy dissipation, M is the mean turbulent kinetic energy of fluctuations and N is the energy associated with pressure and kinetic energy diffusion in the normal direction. The angular brackets, $\langle\ \rangle$, denotes the ensemble average. Eliminating q between equations (9) and (50) one obtains

$$\varepsilon = g\beta\left[\phi(L-Y) - \frac{\nu}{\sigma}\frac{d\bar{T}}{dy}\right] + \nu\frac{d^2M}{dy^2} + \frac{dN}{dy}. \tag{52}$$

Equation (52) is analysed in the three layers. The scale of velocity fluctuations in any of the layers is taken as the respective velocity scale of that layer. The scale of pressure fluctuations is taken as the square of the velocity fluctuations in that layer.

4.1. *Inner layer*

Adopting the scales of velocity and pressure fluctuations explained above, the inner variables may be written as

$$\varepsilon = \varepsilon_i(\eta)u_i^4/\nu, \quad M = u_i^2 M_i(\eta), \quad N = u_i^3 N_i(\eta). \tag{53}$$

Introducing equations (14) and (53) in equation (52) one obtains

$$\varepsilon_i = 1 + \frac{d}{d\eta}\left(N_i - \sigma^{-1}f + \frac{dM_i}{d\eta}\right) - Re^{-3/4}\eta. \tag{54}$$

The inner expansions for the variables are

$$\varepsilon_i = \varepsilon_{i1}(\eta) + \lambda_i\varepsilon_{i2}(\eta) + ,\ldots,$$
$$M_i = M_{i1}(\eta) + \lambda_i M_{i2}(\eta) + ,\ldots,$$
$$N_i = N_{i1}(\eta) + \lambda_i N_{i2}(\eta) + ,\ldots, \tag{55}$$

where λ_i is a gauge function. The equation connecting the lowest order terms is

$$\varepsilon_{i1} = 1 + \frac{d}{d\eta}\left(N_{i1} - \sigma^{-1}f_1 + \frac{dM_{i1}}{d\eta}\right). \tag{56}$$

4.2. *Outer layer*

The outer variables are non-dimensionalized with respect to the outer length and velocity scales as

$$\varepsilon = \varepsilon_o(Y)u_o^3/L, \quad M = u_o^2 M_o(Y), \quad N = u_o^3 N_o(Y). \tag{57}$$

In terms of variables (20) and (57), equation (52) can be written as

$$\varepsilon_o = 1 - Y + \frac{dN_o}{dY} - Re^{-1}\left(\sigma^{-1}\frac{dF}{dY} - \frac{d^2M_o}{dY^2}\right). \tag{58}$$

The outer expansions are

$$\varepsilon_o = \varepsilon_{o1}(Y) + \lambda_o\varepsilon_{o2}(Y) + ,\ldots,$$
$$M_o = M_{o1}(Y) + \lambda_o M_{o2}(Y) + ,\ldots,$$
$$N = N_{o1}(Y) + \lambda_o N_{o2}(Y) + ,\ldots, \tag{59}$$

where λ_o is the gauge function. The equations satisfied by the lowest order terms are

$$\varepsilon_{o1} = 1 - Y + \frac{dN_{o1}}{dY}, \tag{60}$$

showing that the effects of molecular viscosity are of higher order.

4.3. *Intermediate layer*

In this layer non-dimensional variables are

$$\varepsilon = \varepsilon_\delta(\zeta)u_\delta^3/\delta, \quad M = u_\delta^2 M_\delta(\zeta), \quad N = u_\delta^3 N_\delta(\zeta). \tag{61}$$

Equation (52), using equations (26) and (61) produces

$$\varepsilon_\delta = 1 + \frac{dN_\delta}{d\zeta} - Re^{-3/7}\left(\zeta + \sigma^{-1}\frac{d\mathscr{F}}{d\zeta} - \frac{d^2M_\delta}{d\zeta^2}\right). \tag{62}$$

The intermediate expansions are

$$\varepsilon_\delta = \varepsilon_{\delta1}(\zeta) + Re^{-3/7}\varepsilon_{\delta2}(\zeta) + ,\ldots,$$
$$M_\delta = M_{\delta1}(\zeta) + O(1),$$
$$N_\delta = N_{\delta1}(\zeta) + Re^{-3/7}N_{\delta2}(\zeta) + ,\ldots,. \tag{63}$$

The equations connecting these terms are

$$\varepsilon_{\delta1} = 1 + \frac{dN_{\delta1}}{d\zeta}, \tag{64}$$

$$\varepsilon_{\delta2} = -\zeta - \frac{1}{\sigma}\frac{d\mathscr{F}_1}{d\zeta} + \frac{d^2M_{\delta1}}{d\zeta^2} + \frac{dN_{\delta2}}{d\zeta}. \tag{64b}$$

4.4. *Matching*

The behaviour of ε is the main interest. Having matched $\bar{T}$ in Section 3 the matching of M and N only needs to be considered. The matching of the inner and intermediate layer leads to

$$M_{i1} \sim \alpha\eta^{1/3}, \quad \eta\to\infty,$$
$$M_{\delta1} \sim \alpha\zeta^{1/3}, \quad \zeta\to 0,$$
$$N_{i1} \sim \beta_1\eta, \quad N_{i2} \sim \beta_2\eta, \quad \eta\to\infty,$$
$$N_{\delta1} \sim \beta_1\zeta, \quad N_{\delta2} \sim \beta_2\zeta, \quad \zeta\to 0,$$

where α, β_1 and β_2 are constants. Based on the above relations the expressions for ε in the overlap region O_i are

$$\varepsilon_{i1} = 1+\beta_1-\eta^{-4/3}\left(\frac{A_i}{3\sigma}-\frac{2\alpha}{9}\right), \quad \eta\to\infty, \quad (65)$$

$$\varepsilon_{\delta 1} = 1+\beta_1, \quad \zeta\to 0, \quad (66a)$$

$$\varepsilon_{\delta 2} = -\zeta-\zeta^{-4/3}\left(\frac{A_i}{3\sigma}-\frac{2\alpha}{9}\right)+\beta_2, \quad \zeta\to 0. \quad (66b)$$

Likewise, the matching of the intermediate and outer layer leads to

$$M_{\delta 1} = \tilde{\alpha}\zeta^{1/3}, \quad \zeta\to\infty,$$
$$M_{o1} = \tilde{\alpha}Y^{1/3}, \quad Y\to 0,$$
$$N_{\delta 1} = \tilde{\beta}_1\zeta, \quad N_{\delta 2}\sim\tilde{\beta}_2\zeta, \quad \zeta\to\infty,$$
$$N_{o1} = \tilde{\beta}_1 Y, \quad N_{o2}\sim\beta_2 Y, \quad Y\to 0,$$

and the expressions for ε in overlap domain O_o are

$$\varepsilon_{o1} = 1+\tilde{\beta}_1-Y+O(Re^{-1}), \quad Y\to 0, \quad (67)$$

$$\varepsilon_{\delta 1} = 1+\tilde{\beta}_1, \quad \zeta\to\infty, \quad (68a)$$

$$\varepsilon_{\delta 2} = -\zeta-\zeta^{-4/3}\left(\frac{A_o}{3\sigma}-\frac{2\tilde{\alpha}}{9}\right)+\tilde{\beta}_2, \quad \zeta\to\infty. \quad (68b)$$

5. RESULTS AND DISCUSSION

The results of the temperature profile may be summarized as

inner layer:

$$\bar{T} = T_w+t_i f_1(\eta)+,\ldots,$$
$$\eta\to\infty, \quad f_1\sim -A_i\eta^{-1/3}+B; \quad (69)$$

intermediate layer:

$$\bar{T} = T_r+t_\delta\mathscr{F}_1(\zeta)+,\ldots,$$
$$\zeta\to 0, \quad \mathscr{F}_1\sim -A_i\zeta^{-1/3}+C, \quad (70a)$$
$$\zeta\to\infty, \quad \mathscr{F}_1\sim -A_o\zeta^{-1/3}+\tilde{C}; \quad (70b)$$

outer layer:

$$\bar{T} = T_o+t_o F_1(Y)+,\ldots,$$
$$Y\to 0, \quad F_1\sim -A_o Y^{-1/3}+D. \quad (71)$$

In the three-layer analysis there are two overlap domains, O_i and O_o, where the slopes, A_i and A_o, of the $-1/3$ power law for the temperature distribution could possibly be different. From the available measurements of the temperature profile it is not possible to ascertain whether A_i is different from A_o. Therefore, one considers the case

$$A_i = A_o = A. \quad (72)$$

Based on equation (72) the relations (69) and (71) are the classical laws in the inner and outer layers. In the intermediate layer, expressions (70a) and (70b) show that the temperature distribution is also governed by the $-1/3$ power laws, whose intercepts C and $\tilde{C}$ in O_i and O_o, could possibly be different. Further, relation (72) also implies the matchability of the inner and outer solutions (69) and (71). In such a situation the intermediate layer forms the matching region between the inner and outer layers, thereby implying that

$$C = \tilde{C}. \quad (73)$$

The results for the Reynolds heat flux distribution for this case are

inner layer:

$$q/\phi L = h_1(\eta)+,\ldots,$$
$$\eta\to\infty, \quad h_1\sim 1-\frac{1}{3\sigma}A\eta^{-4/3}; \quad (74)$$

intermediate layer:

$$q/\phi L = 1-Re^{-3/7}\mathscr{H}_1(\zeta)+,\ldots,$$
$$\mathscr{H}_1\sim 1-Re^{-3/7}\left(\zeta+\frac{1}{3\sigma}A\zeta^{-4/3}\right); \quad (75)$$

outer layer:

$$q/\phi L = H_1(Y)+,\ldots,$$
$$H_1\sim 1-Y, \quad Y\to 0. \quad (76)$$

To the lowest order, the inner and outer solutions, relations (74) and (76), for Reynolds heat flux also matches directly to yield the value of unity. Based on relations (74) and (76) the existence of the intermediate layer can as well be demonstrated if, following Afzal [20], in the intermediate layer one postulates that the excess of the Reynolds heat flux over the wall heat flux, from inner and outer solutions, relations (74) and (76), are of the same order. The departure of Reynolds heat flux over the wall heat flux $A/(3\sigma\eta^{4/3})$ in the inner solution, relation (74), is due to the molecular transport whereas Y in the outer layer, relation (76), is due to volumetric heat generation. The molecular conduction and volumetric heat generation contributions are of same order for

$$\zeta = Re^{-4/7}\eta = YRe^{3/7}, \quad (77)$$

fixed as $Re\to\infty$. This is the definition of the intermediate layer described in Section 3.

It may be seen that expression (75) for Reynolds stress in the intermediate layer is associated with a maximum whose magnitude, q_m, and location, ζ_m, are given by

$$q_m/\phi L = 1-Re^{-3/7}\left[\frac{7}{4}\left(\frac{4A}{9\sigma}\right)^{3/7}\right], \quad \zeta_m = \left(\frac{4A}{9\sigma}\right)^{3/7}. \quad (78)$$

For the turbulent energy dissipation, ε, the results in the three layers based on equation (72) are

inner:

$$\varepsilon\nu/u_i^4 = \varepsilon_{i1}(\eta)+O(Re^{-3/4})$$
$$\sim 1+\beta_1-\eta^{-4/3}\left(\frac{A}{3\sigma}-\frac{2\alpha}{9}\right), \quad \eta\to\infty; \quad (79)$$

intermediate:

$$\varepsilon\delta/u_\delta^3 = \varepsilon_{\delta 1}(\zeta)+Re^{-3/7}\varepsilon_{\delta 2}(\zeta)+,\ldots,$$

$$\sim 1+\beta_1-Re^{-3/7}\left[\zeta+\zeta^{-4/3}\left(\frac{A}{3\sigma}-\frac{2\alpha}{9}\right)-\beta_2\right]; \quad (80)$$

outer:

$$\varepsilon L/u_o^3 = \varepsilon_{o1}(Y)+O(Re^{-1}), \sim 1+\beta_1-Y, \quad Y\to 0. \quad (81)$$

It may be seen from relation (80) that the turbulent energy dissipation in the intermediate layer is also associated with a maximum whose magnitude, ε_m, and location, $\zeta_{m\varepsilon}$, are given by

$$\varepsilon_m = g\beta\phi L\left[1+\beta_1-Re^{-3/7}\left\{\frac{7}{4}\left(\frac{4A}{9\sigma}-\frac{8}{27}\right)^{3/7}-\beta_2\right\}\right], \quad (82a)$$

$$\zeta_{m\varepsilon} = \left(\frac{4A}{9\sigma}-\frac{8\alpha}{27}\right)^{3/7}. \quad (82b)$$

Based on equation (72), the heat transfer law, equation (48), reduces to

$$Nu = Ra_I^{1/4}/[\hat{B}(\sigma)-D\sigma^{-1/3}Ra_I^{-1/12}], \quad (83)$$

where $\hat{B}(\sigma) = B(\sigma)\sigma^{-1/2}$ and σ is of order unity. The heat transfer relation (83) is similar to relation (52) of Cheung [17] where data [10, 11] are used to determine B and D. It is worthwhile to note that in the present relation (83) D is independent of σ whereas in relation (52) of ref. [17] B_2 (equivalent to D) depends on σ, when σ is of order unity. This dependence of B_2 on σ cannot be justified on physical grounds as well. The present work and ref. [17] show that D (or B_2) is associated with the outer layer solution. If D (or B_2) is to depend on σ, then it implies that the molecular transport processes in the outer layer are important and the flow cannot be regarded as a fully developed turbulence, rather it should be transitional. A fully developed turbulence demands that the outer layer should be independent of molecular transport processes [13, 14].

From the above results it follows that for the lowest order the classical two-layer theory should suffice and to the next order intermediate layer is needed and a three-layer picture should be considered. The higher order effect to the present analysis can also be estimated. In the inner layer, equation (16), the higher order effects are of order $Re^{-3/4}$ whereas in the outer layer, equation (22), they are of order Re^{-1}. On the other hand the intermediate layer suggests that the higher order effect of order $Re^{-3/7}$ should be considered first. Therefore, in the asymptotic expansions for the inner layer, equations (18a) and (18b), intermediate layer, equations (31a) and (31b), and outer layer, equations (23a) and (23b), one choose

$$e_1 = e_2 = e_3 = Re^{-3/7}. \quad (84)$$

The matching of the higher order (terms in the asymptotic expansions for three layers) is carried out and the result for the heat transfer law is

$$Nu = Ra_I^{1/4}/[\hat{B}(\sigma)-D\sigma^{-1/3}Ra_I^{-1/12}-E(\sigma)Ra_I^{-1/7}]. \quad (85)$$

The heat transfer data [10, 11] may be fitted by relation (85) to determine the coefficient E. However, the scatter in the data (see Figs. 1 and 4 of ref. [17]) is such that it is not possible to determine the value of E, unless better data is available.

It may not be out of place to consider the general implications of the present work on Rayleigh–Bénard turbulent convection (between the two horizontal plates heated differentially), as the two problems belong to the same class of flows. Some of the results obtained here are also valid for Rayleigh–Bénard convection provided Ω is replaced by the heat flux, q_w, from the plate under consideration. For this case expression (85) for the Nusselt number may be written as

$$Nu = Ra^{1/3}/[\hat{B}-D\sigma^{-1/3}(Nu\,Ra)^{-1/12} + E(Nu\,Ra)^{-1/7}]^{4/3}, \quad (86)$$

where Ra is the Rayleigh number defined as

$$Ra = \sigma g\beta(T_w-T_o)L^3/\nu^2 = Ra_I/Nu. \quad (87)$$

For $E = 0$ expression (86) reduces to that of Long [12]. The length scale, δ, of the intermediate layer is

$$\delta/L = (\nu/u_\delta L)^{-1/2}, \quad (88)$$

$$= Re^{-3/7}. \quad (89)$$

Long and Chen [18] by postulating an analogy with forced convection flow in a pipe have proposed that the intermediate length scale, δ, for natural convection is

$$\delta/L = Re^{-1/2}. \quad (90)$$

It is instructive to compare these expressions with the thickness of intermediate layer, Δ, in the pipe of radius a [19, 21], given by

$$\Delta/a = \left(\frac{\nu}{au_\tau}\right)^{1/2}. \quad (91)$$

In the pipe flow the velocity scale in all the three layers (inner, intermediate and outer) is the friction velocity, u_τ [20, 21]. On the other hand in Rayleigh–Bénard convection the inner and outer scales, u_i and u_o, are different [12]. Therefore, the analogy between buoyant and forced flows cannot be as simple as postulated by Long and Chen [19]. The present result, equation (87), shows that the thickness of the intermediate layer be based on u_δ, the intermediate velocity scale, whereas Long and Chen [19] have based it on the outer velocity scale, u_o.

In the classical two-layer theory of turbulent flows, it is well known that either of the inner or outer layer depends on the local variable associated with that layer [13, 14]. For example, in the shear flow turbulence, the law of the wall depends on the wall (local) variables, ν and u_τ [13, 14]. Further, in Rayleigh–Bénard convection, the inner variables, ν, σ and q_w [12] and

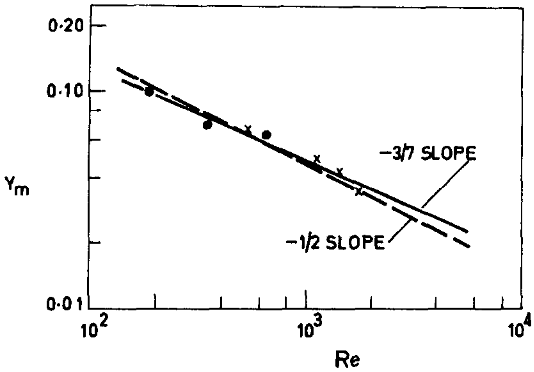

FIG. 1. The location of maxima Y_m in the tangential R.M.S. velocity profile. ●, Deardorff and Willis; ×, Ferreira; ——, −3/7 slope line; ---, −1/2 slope line of Long and Chen.

outer variables, L and q_w [12], are again the local variables. Likewise, the intermediate layer should also depend on its own local variables. The present −3/7 power law conforms to this principle, whereas the −1/2 power law of Long and Chen [19] based on outer velocity scale does not.

A comparison of expressions (89) and (90) with experimental data is presented in Figs. 1 and 2. Figure 1 shows the location of the maxima of tangential R.M.S. velocity in the thermal convection data of Deardorff and Willis [22] and Ferreira [23]. The location Y_l of the levelling off point of the mean temperature profile in the thermal convection data of Deardorff and Willis [22], Sommerscales and Gazda [24] and Thomas and Townsend [25], displayed in Figs. 7 and 8 of ref. [19], are shown here in Figs. 2(a) and (b) along with −3/7 and −1/2 slope lines. Although, the scatter in the data is too large to distinguish −3/7 from −1/2, the fact remains that the present analysis is more rational and removes some of the inconsistencies of the classical theory.

6. CONCLUSIONS

(1) For turbulent natural convection in a horizontal fluid layer, there is an intermediate layer in between the classical inner and outer layers. The length scale of the intermediate layer is $L\,Re^{-3/7}$ rather than $L\,Re^{-1/2}$ proposed by Long and Chen [19].

(2) In the intermediate layer, to the lowest order, the Reynolds heat flux gradient is dominant and Reynolds heat flux is equal to the heat flux at the surface. To the

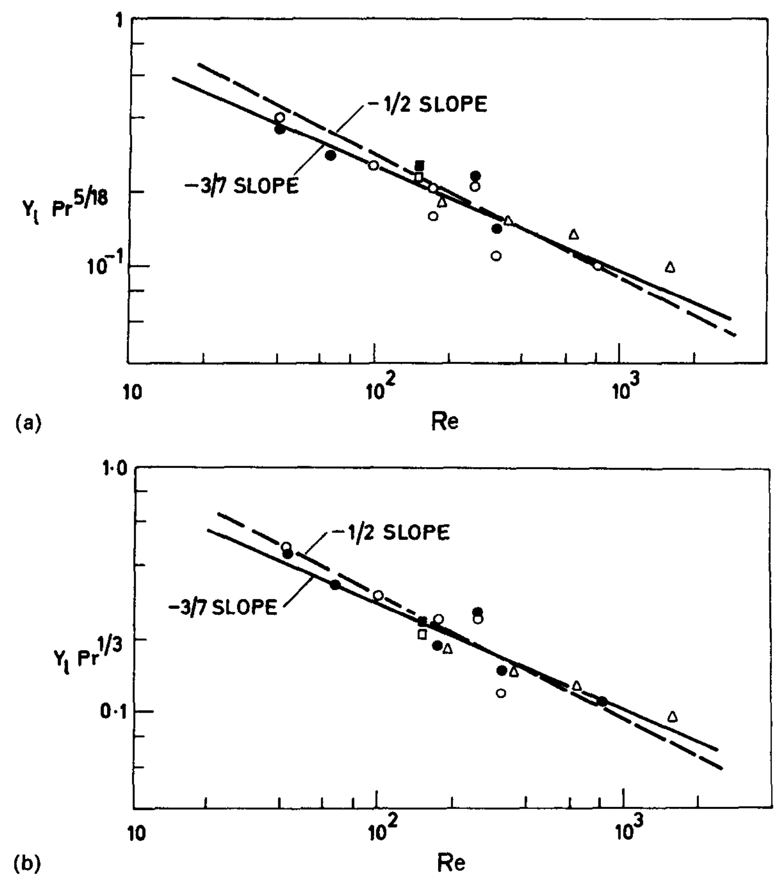

FIG. 2. The location of the 'levelling off' point Y_l of mean temperature profile in turbulent thermal convection between horizontal plates. △, Deardorff and Willis; ○, Sommerscales and Gazda; □, Thomas and Townsend. Open symbols, hot wall; solid symbols, cold wall. ——, −3/7 slope line; ---, −1/2 slope line of Long and Chen. (a) For low Prandtl number. (b) For high Prandtl number.

next order molecular conduction, volumetric heat generation and perturbed Reynolds heat flux are of the same order.

(3) The intermediate layer is matched with the inner and outer layers leading to two overlap domains, O_i and O_o, where slopes A_i and A_o of the $-1/3$ power law for temperature could possibly be different. From the available measurements it is not possible to ascertain whether A_i is different from A_o.

(4) Under the condition $A_i = A_o$ the lowest order results for temperature, Reynolds heat flux and heat transfer law are the same as given by the two-layer classical theory. Thus to the lowest order, the intermediate layer forms the matching domain between the inner and outer layers. There is no need to consider the intermediate layer separately and the two-layer classical theory should suffice.

(5) To the next order, the intermediate layer is a distinguished limit and a three-layer picture is needed. The three-layer theory leads to prediction of maxima in Reynolds heat flux and in turbulent energy dissipation. To this order, the heat transfer law is independent of whether a three- or two-layer picture is considered.

(6) Further, higher order terms in the asymptotic expansions in the three (inner, intermediate and outer) layers lead to a more general heat transfer law. The scatter in the available data is such that it is not possible to numerically determine this higher order effect and there is a need for better data.

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CONVECTION NATURELLE TURBULENTE DANS UNE COUCHE FLUIDE HORIZONTALE AVEC SOURCES D'ENERGIE VOLUMIQUES: UNE COUCHE INTERMEDIAIRE

Résumé—La convection naturelle turbulente avec sources thermiques dans une couche fluide limitée par une plaque inférieure adiabatique et une plaque supérieure isotherme est étudiée à de grandes valeurs du nombre de Rayleigh interne. L'existence d'une couche intermédiaire est identifiée dont l'échelle de longueur est $Re^{-3/7}$ (Re est un nombre de Reynolds basé sur la vitesse caractéristique externe et les échelles de longueur) plutôt que $Re^{-1/2}$ comme proposé par Long et Chen. Les maxima des flux thermiques de Reynolds et de dissipation turbulente d'énergie sont associés à la couche intermédiaire. On montre qu'à l'ordre inférieur, la théorie classique à deux couches peut suffire et qu'à l'ordre suivant, il faut une couche intermédiaire.

TURBULENTE FREIE KONVEKTION IN EINER HORIZONTALEN FLÜSSIGKEITSSCHICHT MIT VOLUMETRISCHEN ENERGIEQUELLEN: EINE ZWISCHENSCHICHT

Zusammenfassung—Die turbulente, natürliche Konvektion aufgrund von Wärmequellen in einer horizontalen Flüssigkeitsschicht, die nach unten durch eine adiabate und nach oben durch eine isotherme Platte begrenzt ist, wurde bei grossen inneren Rayleigh-Zahlen untersucht. Die Existenz einer Zwischenschicht wurde festgestellt, deren Längenmaßstab eher von $Re^{-3/7}$ (Re ist eine Reynolds-Zahl, die mit den äusseren charakteristischen Geschwindigkeits- und Längenmaßstäben gebildet ist) als von $Re^{-1/2}$ wie von Long und Chen vorgeschlagen, abhängt. Die Maxima des Reynoldschen Wärmeflusses und der turbulenten Energie-Dissipation hängen mit der Zwischenschicht zusammen. Die Zwischenschicht ist an die innere und die äussere Schicht angepasst. Es wird angezeigt, dass für die niedrigste Ordnung die klassische Zwischenschichttheorie genügen sollte und für die nächste Ordnung eine Zwischenschicht benötigt wird.

ТУРБУЛЕНТНАЯ ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ В ГОРИЗОНТАЛЬНОМ СЛОЕ ЖИДКОСТИ С ВНУТРЕННИМИ ИСТОЧНИКАМИ ЭНЕРГИИ. ПРОМЕЖУТОЧНЫЙ СЛОЙ

Аннотация—При больших значениях числа Рэлея для жидкости проведено исследование индуцируемой тепловым источником турбулентной естественной конвекции в горизонтальном слое жидкости, ограниченном снизу адиабатической, а сверху изотермической пластинами. Выявлено существование промежуточного слоя, толщина которого равна $Re^{-3/7}$(Re—число Рейнольдса, построенное с помощью внешних характерных масштабов скорости и длины), а не $Re^{-1/2}$, как утверждают Лонг и Чен. Максимумы теплового потока, зависящего от числа Рейнольдса, и турбулентной диссипации энергии расположены в этом промежуточном слое. Промежуточный слой сшивается с внутренним и внешним слоями. Показано, что для расчета с точностью до членов низшего порядка достаточно классической двухслойной теории, а для следующего порядка требуется учет промежуточного слоя.